

USACH-FM-01/11

Nonlinear holomorphic supersymmetry, Dolan-Grady relations and Onsager algebra

Sergey M. Klishevich^{a,b*} and Mikhail S. Plyushchay^{a,b†}

^a*Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile*

^b*Institute for High Energy Physics, Protvino, Russia*

Abstract

Recently, it was noticed by us that the nonlinear holomorphic supersymmetry of order $n \in \mathbb{N}$, $n > 1$, (n -HSUSY) has an algebraic origin. We show that the Onsager algebra underlies n -HSUSY and investigate the structure of the former in the context of the latter. A new infinite set of mutually commuting charges is found which, unlike those from the Dolan-Grady set, include the terms quadratic in the Onsager algebra generators. This allows us to find the general form of the superalgebra of n -HSUSY and fix it explicitly for the cases of $n = 2, 3, 4, 5, 6$. The similar results are obtained for a new, contracted form of the Onsager algebra generated via the contracted Dolan-Grady relations. As an application, the algebraic structure of the known 1D and 2D systems with n -HSUSY is clarified and a generalization of the construction to the case of nonlinear pseudo-supersymmetry is proposed. Such a generalization is discussed in application to some integrable spin models and with its help we obtain a family of quasi-exactly solvable systems appearing in the PT -symmetric quantum mechanics.

*E-mail: sklishev@lauca.usach.cl

†E-mail: mplyushc@lauca.usach.cl

1 Introduction

Recently, there appeared a significant interest in supersymmetry characterized by a nonlinear superalgebra [1, 2, 3, 4, 5, 6, 7]. It is a natural generalization of the usual supersymmetry given by a linear superalgebra [8, 9]. The nonlinear supersymmetry emerges, in particular, in the context of quantization of the one-dimensional mechanical systems possessing supersymmetry of a general (linear or nonlinear) form [4]. It was proved [4] that any classical 1D supersymmetric system is symplectomorphic to a supersymmetric system of a canonical form in which the supercharges are (anti)holomorphic monomials of oscillator-like even and odd variables. The anomaly-free quantization of a canonical system with (anti)holomorphic supercharges is possible only for the oscillator-like bosonic operators of the special restricted form. As a result [4], the anomaly-free quantization leads to appearance of the family of the 1D quantum systems with nonlinear holomorphic supersymmetry of order $n \in \mathbb{N}$, $n > 1$, (n -HSUSY) which turns out to be directly related to the so called quasi-exactly solvable systems [10, 11, 12, 13, 14, 15]. The nonlinear holomorphic supersymmetry of the 1D quantum systems [4] was generalized subsequently by us to the 2D case represented by the system of a charged particle on the plane with magnetic field [6]. It was found that the main difference of the 2D systems with holomorphic supersymmetry from the 1D case [4] consists in appearance of an even central charge entering non-trivially into the nonlinear superalgebra. It was also observed [6] that the reduction of the 2D systems with holomorphic supersymmetry can produce nonlinear supersymmetry of a non-holomorphic form [5, 16] related to some families of quasi-exactly solvable systems.

In Ref. [6] we noted that the Hamiltonian of a supersymmetric system and corresponding supercharges can be realised in terms of two mutually conjugate operators Z and \bar{Z} satisfying some nonlinear relations (see Eq. (2.1) below). Such a formulation does not depend on the choice of a particular representation for Z and \bar{Z} . This means that n -HSUSY is based on a universal algebraic foundation [6, 17]. *The present paper is devoted to the detailed investigation of the universal algebraic construction underlying the nonlinear holomorphic supersymmetry.*

The paper is organized as follows. In Section 2 we describe shortly the structure of n -HSUSY revealed in Ref. [6] and discuss the nonlinear algebraic relations underlying it. The nonlinear relations turn out to be a peculiar form of the Dolan-Grady relations which appeared earlier in the context of integrable models. Two different cases of the nonlinear relations have to be distinguished. More simple case generating the contracted Onsager algebra is investigated in Section 3. To our knowledge, such a form of the Onsager algebra was not discussed earlier in literature. We find an infinite set of the mutually commuting operators being quadratic in generators of the contracted Onsager algebra. It is this set of quadratic operators that is associated with the Hamiltonian and central elements of the nonlinear superalgebra. The obtained results are applied for finding the exact form of the nonlinear superalgebra corresponding to the case of n -HSUSY associated with the contracted Onsager algebra. In Section 4 we investigate in the same line the non-contracted case of the Onsager algebra and associated n -HSUSY. In Section 5 we discuss the 1D and 2D systems with nonlinear holomorphic supersymmetry within the general algebraic framework developed in the previous two sections. A minimal modification of the general algebraic construction allows us to obtain the nonlinear pseudo-supersymmetry which can be considered

as a natural generalization of the linear pseudo-supersymmetry [18]. In this way in section 6 we treat some integrable spin models and obtain a family of quasi-exactly solvable systems appearing in the context of the PT -symmetric quantum mechanics [19]. In section 7 the brief summary of the obtained results is presented.

2 The structure of n -HSUSY and Onsager algebra

The holomorphic supersymmetry is constructed in terms of the pairs of even, Z , $\bar{Z} = Z^\dagger$, and odd, θ^+ , $\theta^- = (\theta^+)^\dagger$, operators. The odd operators commute with the even ones and satisfy the anticommutation relations: $\{\theta^-, \theta^+\} = 1$, $(\theta^\pm)^2 = 0$. If the even operators obey the nonlinear relations

$$[Z, [Z, [Z, \bar{Z}]]] = \omega^2 [Z, \bar{Z}], \quad [\bar{Z}, [\bar{Z}, [Z, \bar{Z}]]] = \bar{\omega}^2 [Z, \bar{Z}] \quad (2.1)$$

with $\omega \in \mathbb{C}$, $\bar{\omega} = \omega^*$, then for the Hamiltonian of the form

$$H_n = \frac{1}{2} \{ \bar{Z}, Z \} + \frac{n}{2} [Z, \bar{Z}] \cdot [\theta^+, \theta^-] \quad (2.2)$$

the odd nilpotent operators defined by the recurrent relations

$$\begin{aligned} Q_{n+2} &= \left(Z^2 - \left(\frac{n+1}{2} \right)^2 \omega^2 \right) Q_n, & Q_0 &= \theta^+, & Q_1 &= Z\theta^+, \\ \bar{Q}_{n+2} &= \left(\bar{Z}^2 - \left(\frac{n+1}{2} \right)^2 \bar{\omega}^2 \right) \bar{Q}_n, & \bar{Q}_0 &= \theta^-, & \bar{Q}_1 &= \bar{Z}\theta^-, \end{aligned} \quad (2.3)$$

are the integrals of motion, $[Q_n, H_n] = [\bar{Q}_n, H_n] = 0$, being the supercharges of the system (2.2) [6]. In what follows, we will refer to the system with the Hamiltonian H_n and the supercharges Q_n , \bar{Q}_n defined by Eqs. (2.1)–(2.3) as to the nonlinear holomorphic supersymmetric system.

Relations (2.1) can be treated as “integrability conditions” of n -HSUSY. For the linear case $n = 1$, unlike the nonlinear one, $n > 1$, the restrictions (2.1) on the operators Z , \bar{Z} are not necessary since the supersymmetry,

$$[Q_1, H_1] = 0, \quad [\bar{Q}_1, H_1] = 0, \quad Q_1^2 = \bar{Q}_1^2 = 0,$$

appears just due to the special form of the Hamiltonian. Hence, the $n = 1$ supersymmetry has a clear algebraic origin different from that of the nonlinear case $n > 1$. Let us note also that such a linear supersymmetry is always holomorphic by the construction. In this sense, the nonlinear holomorphic supersymmetry is a natural algebraic generalization of the linear supersymmetry.

In the 1D case [4], n -HSUSY is obtained via the realization $Z = \frac{d}{dx} + W(x)$, $\bar{Z} = -\frac{d}{dx} + W(x)$, where $W(x)$ is a real function (superpotential) defined by the relations (2.1). In this representation the supercharges and the Hamiltonian of one-dimensional system with n -HSUSY form a polynomial superalgebra, $\{Q_n, \bar{Q}_n\} = P_n(H_n)$, where P_n is the n th degree polynomial. The supercharges, the Hamiltonian and the even central charge appearing in the 2D plane case [6] also form a nonlinear superalgebra of the polynomial type, but there

the anticommutator of the supercharges produces a polynomial in both even integrals of motion. As we shall see, generally a system with n -HSUSY admits an infinite number of the central charges. They enter nontrivially into the superalgebra via the anticommutator of the supercharges being a polynomial in the Hamiltonian as well as in the central charges. However, depending on the concrete representation of the operators Z and \bar{Z} , some of the central charges can turn into zero, or can be representable in terms of other central charges. Note that the polynomiality of the superalgebra makes n -HSUSY to be similar to the Yangian and finite W-algebras [20, 21, 22].

The amazing fact is that the “integrability conditions” (2.1) are, in fact, the well-known Dolan-Grady relations [23]. For the first time such relations appeared in the context of integrable models [23]. In Ref. [23] it was proved that relations of the form (2.1) allow ones to construct the infinite number of commuting operators. It is such a set of mutually commuting operators that relates naturally a construction to integrable models. In the most elegant way the construction can be represented in terms of the Onsager algebra (\mathfrak{OA}), which is recursively generated by means of the Dolan-Grady relations [23]. The Onsager algebra [24] is spanned by the generators A_n and G_n having the commutation relations

$$[A_m, A_n] = 4G_{m-n}, \quad [G_m, A_n] = 2A_{n+m} - 2A_{n-m}, \quad [G_m, G_n] = 0, \quad (2.4)$$

where $m, n \in \mathbb{Z}$. In fact, the operators A_0 and A_1 are the generating elements of \mathfrak{OA} obeying the Dolan-Grady relations while all the other operators are defined recursively by the relations (2.4) [25, 26]. Actually, we have the correspondence $A_0 \sim \bar{Z}$, $A_1 \sim Z$. However, unlike the present case (2.1), the operators A_0 , A_1 are usually supposed to be Hermitian. The conventional commutation relations (2.4) correspond to the choice $\omega^2 = 16$, which for $\omega \neq 0$ can always be achieved via the appropriate rescaling of the operators. Though the generating elements of \mathfrak{OA} , A_0 and A_1 , are usually supposed to be Hermitian, they simultaneously are related by some kind of duality [23]. This generates an involution on the whole Onsager algebra to be additional to the Hermitian conjugation.

The Onsager algebra finds numerous applications in studying integrable models [23, 25, 26, 27]. Specifically, the infinite set of the mutually commuting operators constructed by Dolan and Grady can be represented as

$$2J_m = A_m + A_{-m} + \lambda(A_{m+1} + A_{1-m}), \quad (2.5)$$

where $\lambda \in \mathbb{R}$. The commutativity of the operators J_m can easily be verified within the framework of the algebra (2.4) (cf. with the original proof of Ref. [23]). Then, treating the operator $J_0 = A_0 + \lambda A_1$ as a Hamiltonian, one obtains, generally, an integrable system with infinite number of conserved commuting charges. In various models, such as the Baxter eight-vertex [28, 29, 30], the two-dimensional Ising [24] and Z_N spin models [27], there appear the charges of the form (2.5) being equivalent to those which result from the transfer matrix formulation and exact quantum integrability of the system. The power of the algebraic formulation (2.4), (2.5) is that it does not refer to the number of dimensions or to the nature of the space-time manifold which can be lattice, continuum or loop space. However, we should note that till now the application of the construction to dimensions $D > 1$ is an open problem.

As we will see, the set (2.5) is not a unique set of mutually commuting operators that can be constructed in terms of generators of the Onsager algebra. It turns out that there exists

also a set of commuting operators to be quadratic in the generators of \mathfrak{DA} . Such commuting operators naturally appear in the scheme of the nonlinear holomorphic supersymmetry.

3 Contracted Onsager algebra

Before discussing the general case of relations (2.1) with $\omega \neq 0$ and the corresponding Onsager algebra, let us consider a more simple reduced case with $\omega = 0$. Such a construction is also of a significant physical interest. For example, the family of quasi-exactly solvable systems with the x^6 -potential is produced by a reduction of the plane system with n -HSUSY in which $\omega = 0$ [6]. It is worth also noting that n -HSUSY admits a generalization to the systems with nontrivial 2D Riemannian geometry only in this case [31].

The operators $Z_0 \equiv Z$ and $\bar{Z}_0 \equiv \bar{Z}$ together with the contracted ($\omega = \bar{\omega} = 0$) Dolan-Grady relations (2.1) recursively generate the following infinite-dimensional algebra:

$$\begin{aligned} [Z_m, \bar{Z}_n] &= B_{m+n+1}, & [Z_n, B_m] &= Z_{m+n}, & [B_m, \bar{Z}_n] &= \bar{Z}_{m+n}, \\ [Z_m, Z_n] &= 0, & [\bar{Z}_m, \bar{Z}_n] &= 0, & [B_m, B_n] &= 0, \end{aligned} \quad (3.1)$$

where $m, n \in \mathbb{Z}_+$ and $B_0 = 0$ is implied. We denote this contracted Onsager algebra as \mathfrak{COA} and name operators Z_0, \bar{Z}_0 the generating elements of the algebra (3.1).

Algebra \mathfrak{COA} possesses the involution defined by the relations

$$\bar{Z}_m = Z_m^\dagger, \quad B_m^\dagger = B_m. \quad (3.2)$$

Besides, the natural grading is induced in it by putting $\text{gr } Z_0 = \text{gr } \bar{Z}_0 = 1$:

$$\text{gr } Z_m = \text{gr } \bar{Z}_m = 2m + 1, \quad \text{gr } B_m = 2m. \quad (3.3)$$

This grading turns out to be a useful tool in practical calculations, e.g., in computing non-linear superalgebra.

One can verify that the operators Z_k, \bar{Z}_k with $k \in \mathbb{N}$ also obey the contracted Dolan-Grady relations

$$[Z_k, [Z_k, [Z_k, \bar{Z}_k]]] = 0, \quad [\bar{Z}_k, [\bar{Z}_k, [Z_k, \bar{Z}_k]]] = 0. \quad (3.4)$$

This means that one can consider the operators Z_k, \bar{Z}_k as a new pair of generating elements, $Z_0^{(k)}, \bar{Z}_0^{(k)}$, which induce an infinite-dimensional subalgebra $\mathfrak{COA}_k \subset \mathfrak{COA}$. The inclusion $\mathfrak{COA}_k \rightarrow \mathfrak{COA}$ is given by

$$Z_m^{(k)} = Z_{2km+k+m}, \quad \bar{Z}_m^{(k)} = \bar{Z}_{2km+k+m}, \quad B_m^{(k)} = B_{m(2k+1)}. \quad (3.5)$$

From (3.4) it follows that for any $k \in \mathbb{N}$ the algebra \mathfrak{COA}_k itself is a contracted Onsager algebra, i.e. we have the natural isomorphism, $\mathfrak{COA}_k \simeq \mathfrak{COA}$. Obviously, any algebra \mathfrak{COA}_k also has an infinite number of isomorphic subalgebras and so on. This property resembles fractals in the sense that the algebra consists of infinite subsets with the very algebraic properties of the initial set and at any level of embedding we observe the same algebraic structure. Moreover, this property leads to the following corollary. First, we note that the

operators (2.5) commute with each other in the case of the contracted Onsager algebra as well (see the identification (4.1) between the generators A_n , $n \in \mathbb{Z}$, and Z_m , \bar{Z}_m , $m \in \mathbb{Z}_+$, below). Then, for any given representation of the generating elements A_0 , A_1 , besides the main set of the commuting charges (2.5), in general, there exist additional independent sets associated with the sequence of the subalgebras¹ \mathfrak{CDA}_1 , \mathfrak{CDA}_2 , \dots . If a representation of the operators A_0 , A_1 permits to generate the whole infinite-dimensional algebra \mathfrak{CDA} with linearly independent generators, then the number of such sets is infinite.

It is known that the Onsager algebra (2.4) can be regarded as a fixed subalgebra of the sl_2 -loop algebra, $C[t, t^{-1}] \otimes sl_2$ [32]. Such a representation facilitates a systematic study of the algebraic structure of the Onsager algebra [32]. The contracted algebra \mathfrak{CDA} can also be represented as a loop extension of the sl_2 algebra:

$$Z_m = \frac{1}{\sqrt{2}} t^{2m+1} L_-, \quad \bar{Z}_m = \frac{1}{\sqrt{2}} t^{2m+1} L_+, \quad B_m = \frac{1}{2} t^{2m} L_0, \quad (3.6)$$

where $m \in \mathbb{Z}_+$ and operators L_\pm , L_0 are the generators of the sl_2 algebra, $[L_-, L_+] = L_0$, $[L_0, L_\pm] = \pm 2L_\pm$. Note here that the degrees of t in generators of \mathfrak{CDA} coincide with the corresponding grading numbers (3.3). The representation (3.6) implies that the algebra \mathfrak{CDA} is a subalgebra of the loop algebra $C[t] \otimes sl_2$.

Let us consider the following infinite set of operators:

$$J_n^m = \frac{1}{2} \sum_{p=1}^m (\{\bar{Z}_{p-1}, Z_{m-p}\} - B_p B_{m-p}) - \frac{n}{2} B_m, \quad (3.7)$$

where $m \in \mathbb{N}$ and n is an arbitrary real number. Using the commutation relations (3.1), one can demonstrate directly the commutativity of all these operators with the same value of the parameter n ,

$$[J_n^p, J_n^m] = 0. \quad (3.8)$$

So, in addition to the set of commuting operators (2.5) linear in generators of \mathfrak{CDA} , there exists the infinite set of the commuting operators (3.7) quadratic in generators of \mathfrak{CDA} .

Now, we pass over to the construction of n -HSUSY and introduce the operators

$$2\mathcal{J}_n^m = J_n^m \theta^- \theta^+ + J_{-n}^m \theta^+ \theta^-, \quad (3.9)$$

which give a supersymmetric extension of (3.7). Assuming $n \in \mathbb{N}$ in Eq. (3.7), one finds that the operator \mathcal{J}_n^1 coincides with the n -supersymmetric Hamiltonian (2.2). Then the set of even mutually commuting operators (3.9) can be extended by the odd nilpotent operators

$$Q_n = Z_0^n \theta^+, \quad \bar{Q}_n = \bar{Z}_0^n \theta^-. \quad (3.10)$$

Using the algebra (3.1) and the relations

$$[Z_0^n, \bar{Z}_m] = \frac{n(n-1)}{2} Z_{m+1} Z_0^{n-2} + n B_{m+1} Z_0^{n-1}, \quad [Z_0^n, B_m] = n Z_m Z_0^{n-1}$$

¹Since $\mathfrak{CDA}_m \simeq \mathfrak{CDA}$ for any $m \in \mathbb{N}$, such a sequence is defined for every subalgebra \mathfrak{CDA}_m as well, and so on.

following from it, one can demonstrate that the operators (3.9) commute with the odd ones, $[Q_n, \mathcal{J}_n^m] = 0$. This means that the operators (3.10) can be treated as the supercharges of the n -supersymmetric system described by means of the Hamiltonian \mathcal{J}_n^1 , in which the set \mathcal{J}_n^k , $k = 2, 3, \dots$, plays the role of the even integrals of motion. The corresponding nonlinear superalgebra is polynomial and centrally extended. For $n = 2, 3, 4, 5, 6$, the corresponding anticommutators of the supercharges² are

$$\begin{aligned}
\{Q_2, \bar{Q}_2\} &= (\mathcal{J}_2^1)^2 + \frac{1}{2}\mathcal{J}_2^2, \\
\{Q_3, \bar{Q}_3\} &= (\mathcal{J}_3^1)^3 + 2\mathcal{J}_3^1\mathcal{J}_3^2 + \mathcal{J}_3^3, \\
\{Q_4, \bar{Q}_4\} &= (\mathcal{J}_4^1)^4 + 5(\mathcal{J}_4^1)^2\mathcal{J}_4^2 + 6\mathcal{J}_4^1\mathcal{J}_4^3 + \frac{9}{4}(\mathcal{J}_4^2)^2 + \frac{9}{2}\mathcal{J}_4^4, \\
\{Q_5, \bar{Q}_5\} &= (\mathcal{J}_5^1)^5 + 10(\mathcal{J}_5^1)^3\mathcal{J}_5^2 + 21(\mathcal{J}_5^1)^2\mathcal{J}_5^3 + 36\mathcal{J}_5^1\mathcal{J}_5^4 + 16\mathcal{J}_5^1(\mathcal{J}_5^2)^2 \\
&\quad + 24\mathcal{J}_5^2\mathcal{J}_5^3 + 36\mathcal{J}_5^5, \\
\{Q_6, \bar{Q}_6\} &= (\mathcal{J}_6^1)^6 + \frac{35}{2}(\mathcal{J}_6^1)^4\mathcal{J}_6^2 + 56(\mathcal{J}_6^1)^3\mathcal{J}_6^3 + (\mathcal{J}_6^1)^2\left(\frac{259}{4}(\mathcal{J}_6^2)^2 + 162\mathcal{J}_6^4\right) \\
&\quad + \mathcal{J}_6^1(220\mathcal{J}_6^2\mathcal{J}_6^3 + 360\mathcal{J}_6^5) + \frac{225}{8}(\mathcal{J}_6^2)^3 + 225\mathcal{J}_6^2\mathcal{J}_6^4 + 100(\mathcal{J}_6^3)^2 + 450\mathcal{J}_6^6.
\end{aligned} \tag{3.11}$$

From these explicit particular relations one can naturally conjecture that the anticommutator of the corresponding supercharges is proportional to a polynomial in the generators $\mathcal{J}_n^1, \mathcal{J}_n^2, \dots, \mathcal{J}_n^n$ for arbitrary n as well. Note that the grading (3.3) facilitates calculation of the nonlinear superalgebra fixing the general form of the polynomial up to numerical coefficients. However, we have not succeeded in fixing the corresponding coefficients for the general case $n \in \mathbb{N}$.

4 Onsager algebra

Let us consider now the case of the non-contracted ($\omega \neq 0$) Onsager algebra in the context of n -HSUSY. The nonlinear holomorphic supersymmetry is constructed in terms of the generators Z, \bar{Z} obeying the relations (2.1), for which the involution $Z^\dagger = \bar{Z}$ is assumed. It is clear that this operation defined on the generating elements of \mathfrak{DA} induces the involution on the whole Onsager algebra (2.4). We treat this involution as a Hermitian conjugation. To make this involution apparent, it is convenient to adopt the notations

$$A_m = 2\sqrt{2}Z_{m-1}, \quad A_{-n} = -2\sqrt{2}\bar{Z}_n, \quad G_{-m} = 2B_m, \tag{4.1}$$

²For $n = 1$, one has the usual linear superalgebra.

where $m \in \mathbb{N}$, while $n \in \mathbb{Z}_+$. Then the Onsager algebra can be rewritten in the form

$$\begin{aligned}
[Z_n, B_m] &= Z_{n+m} - \alpha^{2m} Z_{n-m} + \alpha^{2n+1} \bar{Z}_{m-n-1}, \\
[B_m, \bar{Z}_n] &= \bar{Z}_{n+m} - \alpha^{2m} \bar{Z}_{n-m} + \alpha^{2n+1} Z_{m-n-1}, \\
[Z_n, Z_{n'}] &= \alpha^{2\min(n,n')+1} B_{n'-n}, \\
[\bar{Z}_n, \bar{Z}_{n'}] &= \alpha^{2\min(n,n')+1} B_{n-n'}, \\
[Z_n, \bar{Z}_{n'}] &= B_{n+n'+1}, \\
[B_m, B_{m'}] &= 0,
\end{aligned} \tag{4.2}$$

where $n, n' \in \mathbb{Z}_+$, $m, m' \in \mathbb{N}$, $\min(n, n') = n$ (or n') for $n > n'$ (for $n' > n$), and we imply that the operators Z_k, \bar{Z}_k with $k < 0$ and B_0 vanish in r.h.s. of (4.2). For the sake of simplicity here we put $\omega^2 = 2\alpha$, $\alpha \in \mathbb{R}$, since we consider the Onsager algebra over \mathbb{C} and the parameter ω can always be chosen in this form by the appropriate rescaling of the generators. Now one can make sure that the algebra \mathfrak{OA} (4.2) possesses the involution defined by the relations (3.2).

Although representation (4.2) is more complicated than the original form (2.4), besides more transparent structure with respect to the conjugation (3.2), it has two further advantages: *a)* in notations (4.1) the contracted algebra (3.1) is reproduced easily in the limit $\alpha \rightarrow 0$; *b)* these notations clarify also the property of self-similarity (“fractal” structure) of the Onsager algebra. Indeed, like the generating elements $Z_0 = Z$, $\bar{Z}_0 = \bar{Z}$, the operators Z_m, \bar{Z}_m , $m \in \mathbb{N}$, obey the Dolan-Grady relations:

$$\begin{aligned}
[Z_m, [Z_m, [Z_m, \bar{Z}_m]]] &= 2\alpha^{2m+1} [Z_m, \bar{Z}_m], \\
[\bar{Z}_m, [\bar{Z}_m, [Z_m, \bar{Z}_m]]] &= 2\alpha^{2m+1} [Z_m, \bar{Z}_m].
\end{aligned} \tag{4.3}$$

This means that one can treat the operators Z_m, \bar{Z}_m , $m \in \mathbb{N}$, as new generating elements, $Z_0^{(m)}, \bar{Z}_0^{(m)}$, producing the infinite-dimensional subalgebra $\mathfrak{OA}_m \subset \mathfrak{OA}$. The inclusion $\mathfrak{OA}_m \rightarrow \mathfrak{OA}$ is given by the same relations (3.5) as for the contracted case. So, \mathfrak{OA} also possesses the infinite set of isomorphic subalgebras, $\mathfrak{OA}_m \simeq \mathfrak{OA}$, $m \in \mathbb{N}$. As a consequence, additional sets of mutually commuting operators are associated with every subalgebra \mathfrak{OA}_m . If, like in the contracted case, the representation of the operators Z_0, \bar{Z}_0 permits to generate the whole infinite-dimensional algebra \mathfrak{OA} with linearly independent generators, then the number of such sets is infinite.

By direct calculation one can verify that as in the contracted case, the operators (3.7) form the infinite set of mutually commuting charges for any value of the parameter n (not only integer) playing the role of the coupling constant. The operator J_n^1 can be treated as the Hamiltonian of the system. Then, to realize the nonlinear holomorphic supersymmetry we have to restrict the parameter n to integer values. The supersymmetric extension of the even charges is defined by Eq. (3.9).

By mathematical induction one can prove that the odd nilpotent operators defined by Eqs. (2.3) commute with all the charges (3.7). Let us sketch the proof. The commutativity for $n = 0$ and $n = 1$ can be verified by the direct calculation. Therefore, one has to prove that $[Q_{n+2}, \mathcal{J}_{n+2}^m] = 0$, $m \in \mathbb{N}$, presuming that the relations $[Q_n, \mathcal{J}_n^m] = 0$ are valid. The last

equality can be represented as

$$[\mathcal{Q}_n, J_n^m] - nB_m \mathcal{Q}_n = 0, \quad (4.4)$$

where $\mathcal{Q}_n = \{Q_n, \theta^-\}$. Using (4.4) and the recursive relations (2.3) defining the supercharges, the similar equality for $n \rightarrow n+2$ can be represented in the form

$$[Z_0^2, J_{-n-2}^m] \left(Z_0^2 - \alpha \frac{(n-1)^2}{2} \right) - \left(Z_0^2 - \alpha \frac{(n+1)^2}{2} \right) [Z_0^2, J_{2-n}^m] = 0. \quad (4.5)$$

The last identity can be verified directly, and this proves that the odd nilpotent operators (2.3) are the true supercharges of the system with the Hamiltonian \mathcal{J}_n^1 . It is clear that the operators \mathcal{J}_n^m , $m = 2, 3, \dots$, are the central charges of the system. The corresponding superalgebra is polynomial and centrally extended. For $n = 2, 3, 4, 5, 6$ the anticommutators of the corresponding supercharges are given by the following relations:

$$\begin{aligned} \{Q_2, \bar{Q}_2\} &= (\mathcal{J}_2^1)^2 + \frac{1}{2} (\mathcal{J}_2^2 + \frac{1}{2} \alpha^2), \\ \{Q_3, \bar{Q}_3\} &= (\mathcal{J}_3^1)^3 + 2\mathcal{J}_3^1 (\mathcal{J}_3^2 + \frac{1}{2} \alpha^2) + \mathcal{J}_3^3, \\ \{Q_4, \bar{Q}_4\} &= (\mathcal{J}_4^1)^4 + 5(\mathcal{J}_4^1)^2 (\mathcal{J}_4^2 + \frac{1}{2} \alpha^2) + 6\mathcal{J}_4^1 \mathcal{J}_4^3 \\ &\quad + \frac{9}{4} (\mathcal{J}_4^2 + \frac{1}{2} \alpha^2)^2 + \frac{9}{2} (\mathcal{J}_4^4 + \alpha^4), \\ \{Q_5, \bar{Q}_5\} &= (\mathcal{J}_5^1)^5 + 10(\mathcal{J}_5^1)^3 (\mathcal{J}_5^2 + \frac{1}{2} \alpha^2) + 21(\mathcal{J}_5^1)^2 \mathcal{J}_5^3 + 36\mathcal{J}_5^1 (\mathcal{J}_5^4 + \alpha^4) \\ &\quad + 16\mathcal{J}_5^1 (\mathcal{J}_5^2 + \frac{1}{2} \alpha^2)^2 + 24 (\mathcal{J}_5^2 + \frac{1}{2} \alpha^2) \mathcal{J}_5^3 + 36\mathcal{J}_5^5, \\ \{Q_6, \bar{Q}_6\} &= (\mathcal{J}_6^1)^6 + \frac{35}{2} (\mathcal{J}_6^1)^4 (\mathcal{J}_6^2 + \frac{1}{2} \alpha^2) + 56 (\mathcal{J}_6^1)^3 \mathcal{J}_6^3 \\ &\quad + (\mathcal{J}_6^1)^2 \left(\frac{259}{4} (\mathcal{J}_6^2 + \frac{1}{2} \alpha^2)^2 + 162 (\mathcal{J}_6^4 + \alpha^4) \right) \\ &\quad + \mathcal{J}_6^1 (220 (\mathcal{J}_6^2 + \frac{1}{2} \alpha^2) \mathcal{J}_6^3 + 360\mathcal{J}_6^5) + \frac{225}{8} (\mathcal{J}_6^2 + \frac{1}{2} \alpha^2)^3 \\ &\quad + 225 (\mathcal{J}_6^2 + \frac{1}{2} \alpha^2) (\mathcal{J}_6^4 + \alpha^4) + 100 (\mathcal{J}_6^3)^2 + 450 (\mathcal{J}_6^6 + \frac{3}{2} \alpha^6). \end{aligned} \quad (4.6)$$

It is worth noting that by the substitution $\mathcal{J}_n^{2k} + \frac{k}{2} \alpha^{2k} \rightarrow \mathcal{J}_n^{2k}$, $k \in \mathbb{N}$, the relations (4.6) are reduced exactly to the relations (3.11) corresponding to the case of the contracted Onsager algebra. One can naturally suppose that the same is valid for any n as well. Presently we cannot explain this fact but let us point out the analogy with the usual linear supersymmetry. In the latter case the arbitrariness in the energy shift is used to represent the superalgebra in the conventional form (without constant term having a sense of the central charge in the supercharges' anticommutator), while in the case (4.6) the shifts of the operators \mathcal{J}_n^{2k} allow ones to represent the superalgebra in the universal form independent from the parameter α . It is necessary to note also that the calculation of the superalgebra (4.6) has essentially facilitated the derivation of the exact form of the even central charges.

5 1D and 2D n -HSUSY from the algebraic viewpoint

As we have noted, the nonlinear holomorphic supersymmetry was discussed for the first time in Ref. [4]. The holomorphic supersymmetry arises naturally in pseudo-classical systems

possessing the supersymmetry of the most general form. The Dolan-Grady relations appeared then as conditions of anomaly-free quantization of such classical systems with *nonlinear* holomorphic supersymmetry. In Ref. [6] the construction was generalized to the case of the quantum mechanics on a plane.

In the present section we discuss these quantum systems with nonlinear holomorphic supersymmetry within the algebraic framework developed above proceeding from the different explicit realizations of the operators Z, \bar{Z} obeying the Dolan-Grady relations.

5.1 1D n -HSUSY

In one-dimensional quantum-mechanical systems the nonlinear holomorphic supersymmetry is generated by the linear differential operators

$$Z_0 = \frac{d}{dx} + W(x), \quad \bar{Z}_0 = -\frac{d}{dx} + W(x) \quad (5.1)$$

given in terms of the superpotential $W(x)$ being a real function [4]. The form of the superpotential is fixed by the Dolan-Grady relations (2.1), and in the non-contracted case ($\omega^2 = 2\alpha \neq 0$) the corresponding solution is

$$W(x) = w_+ e^{\sqrt{2\alpha}x} + w_- e^{-\sqrt{2\alpha}x} + w_0, \quad (5.2)$$

where $w_{\pm}, w_0 \in \mathbb{R}$, are the parameters of the system. In the representation (5.1), (5.2), the generating elements give rise to the following finite-dimensional algebra:

$$\begin{aligned} [Z_0, \bar{Z}_0] &= B_1, & [Z_0, B_1] &= G, & [Z_0, G] &= 2\alpha B_1, \\ [B_1, G] &= 0, & [B_1, \bar{Z}_0] &= G, & [G, \bar{Z}_0] &= 2\alpha B_1, \end{aligned} \quad (5.3)$$

with the generator G satisfying, by definition, the relation $G^\dagger = G$. Formally, this algebra admits the two central elements

$$C = 2\alpha(Z_0 + \bar{Z}_0) - G, \quad I = G^2 - 2\alpha B_1^2,$$

the second from which is the quadratic Casimir operator of the algebra. For any irreducible representation of the algebra (5.3) these operators have to be proportional to the unit operator. In representation (5.1), (5.2), one finds that $C = 4\alpha w_0 \cdot \mathbb{1}$ while $I = 2^6 \alpha^2 w_+ w_- \cdot \mathbb{1}$. Therefore, this representation is irreducible: for $\alpha > 0$ the given representation corresponds to the $iso(1,1)$ algebra, whereas the $e(2)$ algebra is produced for $\alpha < 0$. Indeed, in representation (5.1), (5.2) the corresponding linearly independent nontrivial generators can be represented as $\{i\frac{d}{dx}, e^{\pm x}\}$ ($\alpha = \frac{1}{2}$), or in the form $\{i\frac{d}{dx}, e^{\pm ix}\}$ ($\alpha = -\frac{1}{2}$). The first set produces the Poincaré algebra $iso(1,1)$, while the second generates the Euclidean algebra $e(2)$.

It is interesting to note that though originally the 1D supersymmetric system is defined by the 4-parametric superpotential (5.2), the number of independent continuous parameters is equal to two. Rescaling x , one can reduce the parameter α to ± 1 , whereas a shift in x results in rescaling the parameters w_{\pm} . As a result, the independent combinations of the parameters characteristic to the system possessing n -HSUSY are w_0 and $w_+ w_-$. These parameters define the values of the central elements of the algebra (5.3).

The relations (5.3) give rise to the Onsager algebra (4.2). However, in this case the higher generators are linearly dependent and can be represented as

$$\begin{aligned} Z_{2k} &= \alpha^{2k-1}(kG + \alpha Z_0), & Z_{2k-1} &= \alpha^{2k-2}(kG - \alpha \bar{Z}_0), \\ \bar{Z}_{2k} &= \alpha^{2k-1}(kG + \alpha \bar{Z}_0), & \bar{Z}_{2k-1} &= \alpha^{2k-2}(kG - \alpha Z_0), \\ B_k &= k\alpha^{k-1}B_1, \end{aligned} \quad (5.4)$$

where $k \in \mathbb{N}$. For the generators of this form the mutually commuting operators (3.7) are reduced to

$$J_n^m = m\alpha^{m-1}J_n^1 + \frac{1}{4}C_3^{m+1}\alpha^{m-3}I - \frac{(-1)^m + 1}{16}m\alpha^{m-3}C^2, \quad (5.5)$$

where $C_n^m = \frac{m!}{(m-n)!n!}$, $m \in \mathbb{N}$. As a consequence, the one-dimensional system has only one even independent integral of motion being its Hamiltonian.

In the case of the contracted Dolan-Grady relations ($\alpha = 0$) the superpotential acquires the polynomial form

$$W(x) = w_2x^2 + w_1x + w_0, \quad (5.6)$$

where all the parameters w_0, w_1, w_2 are real and, as a consequence, the nonlinear holomorphic supersymmetry turns out to be spontaneously broken [4] (see, however, the next section). In this case the Onsager algebra does not appear since from (5.4) it follows that all the generators of \mathfrak{DA} vanish in the limit $\alpha \rightarrow 0$ except for Z_0, \bar{Z}_0 and B_1 . Similarly, except for J_n^1 , all the integrals (3.7) vanish as well. Thus, from the point of view of the Onsager algebra this contracted system is trivial. One can verify that the resulting algebra obtained from (5.3) by the contraction $\alpha \rightarrow 0$ is a nilpotent algebra.

5.2 2D n -HSUSY

Representation of the nonlinear holomorphic supersymmetry on a plane supplies us with a new, in comparison with the 1D case, phenomenon of the appearance of nontrivial even central charges. In terms of complex coordinates $z = \frac{1}{2}(x_1 + ix_2)$, $\bar{z} = \frac{1}{2}(x_1 - ix_2)$, the generating elements can be represented as

$$Z_0 = \partial + W(z, \bar{z}), \quad \bar{Z}_0 = -\bar{\partial} + \bar{W}(z, \bar{z}). \quad (5.7)$$

In this representation the Hamiltonian (2.2) corresponds to a charged spin-1/2 particle with gyromagnetic ratio $g = 2n$ moving in an external magnetic field $B = \partial_1 A_2 - \partial_2 A_1$ defined by the gauge vector potential with components $A_1 = \text{Im } W$, $A_2 = \text{Re } W$ [6]. The Dolan-Grady relations give rise to the magnetic field of the form

$$B = w_+ e^{\sqrt{2\alpha}x_1} + w_- e^{-\sqrt{2\alpha}x_1} + w e^{i\sqrt{2\alpha}x_2} + \bar{w} e^{-i\sqrt{2\alpha}x_2} \quad (5.8)$$

with $w_{\pm} \in \mathbb{R}$, $w \in \mathbb{C}$, $\bar{w} = w^*$. Unlike the one-dimensional case, the system defined by the representation (5.7), (5.8) admits in addition to the Hamiltonian (2.2) another independent even integral of motion being a central charge \mathcal{J}_n^2 [6]:

$$2\mathcal{J}_n^2 = -\frac{1}{4}(\omega^2 \bar{Z}^2 + \bar{\omega}^2 Z^2) + \partial B \bar{Z} + \bar{\partial} B Z - B^2 + \frac{n}{2} \bar{\partial} \partial B \sigma_3, \quad (5.9)$$

where $Z = Z_0$, $\bar{Z} = \bar{Z}_0$.

Let us discuss this system in the context of the Onsager algebra. The generating elements in representation (5.7), (5.8) define the finite-dimensional algebra

$$\begin{aligned} [Z_0, \bar{Z}_0] &= B_1, & [Z_0, \bar{G}] &= D, & [G, \bar{Z}_0] &= D, \\ [Z_0, B_1] &= G, & [Z_0, G] &= 2\alpha B_1, & [\bar{G}, \bar{Z}_0] &= 2\alpha B_1, \\ [B_1, \bar{Z}_0] &= \bar{G}, & [Z_0, D] &= 2\alpha \bar{G}, & [D, \bar{Z}_0] &= 2\alpha G, \end{aligned} \quad (5.10)$$

where $\bar{G} = G^\dagger$, $D^\dagger = D$, and B_1 , G , \bar{G} and D commute between themselves. So, in the 2D case we have the algebra with six independent generators. This algebra has the two Casimir operators

$$I_1 = B_1 D - \bar{G} G, \quad I_2 = 4\alpha^2 B_1^2 + D^2 - 2\alpha(G^2 + \bar{G}^2).$$

In representation (5.7), (5.8) they are proportional to the unit operator, $I_1 = 2^5 \alpha (w_+ w_- - \bar{w} w) \cdot \mathbb{I}$, $I_2 = 2^7 \alpha^2 (\bar{w} w + w_+ w_-) \cdot \mathbb{I}$. Therefore, the representation of the algebra (5.10) in this case is irreducible. Then, rescaling the coordinates x_1 , x_2 , one gets the set $\{i\partial_{x_1}, i\partial_{x_2}, e^{\pm i x_1}, e^{\pm i x_2}\}$ as independent generators forming the basis of the algebra (5.10). Therefore, this algebra is $e(2) \oplus iso(1, 1)$ with the Casimir operators to be linear combinations of I_1 and I_2 .

Similarly to the 1D case, the number of independent continuous parameters of the two-dimensional system corresponding to the 5-parametric magnetic field (5.8) is equal to two: using the rescaling and shift of the coordinates, one can fix three parameters. The two independent parameters characteristic to the system with n -HSUSY, $w_+ w_-$ and $\bar{w} w$, are defined by linear combinations of the Casimir operators of the algebra (5.10) (or one can say that they fix the values of the Casimir operators).

The Onsager algebra generated in the representation (5.7), (5.8) is nontrivial. Like in the one-dimensional case, all its higher generators are linearly dependent but non-vanishing. In terms of the generators of the algebra (5.10) they are represented in the form

$$\begin{aligned} Z_{2k} &= \alpha^{2k-1} (k\bar{G} + \alpha Z_0), & Z_{2k-1} &= \alpha^{2k-2} (kG - \alpha \bar{Z}_0), & B_{2k-1} &= (2k-1)\alpha^{2k-2} B_1, \\ \bar{Z}_{2k} &= \alpha^{2k-1} (kG + \alpha \bar{Z}_0), & \bar{Z}_{2k-1} &= \alpha^{2k-2} (k\bar{G} - \alpha Z_0), & B_{2k} &= k\alpha^{2k-2} D, \end{aligned} \quad (5.11)$$

where $k \in \mathbb{N}$. Here the integrals (3.7) are reduced to

$$J_n^{2k+1} = (2k+1)\alpha^{2k} J_n^1 - \frac{1}{4} C_3^{2k+2} \alpha^{2k-2} I_1, \quad J_n^{2k} = k\alpha^{2k-2} J_n^2 - \frac{1}{2} C_3^{k+1} \alpha^{2k-4} I_2,$$

i.e. all the commuting charges are linear functions of the two independent integrals of motion (2.2), (5.9).

In the case of the contracted Dolan-Grady relations ($\alpha = 0$) the magnetic field acquires the polynomial form [6]

$$B = w_2 \bar{z} z + \bar{w} z + w \bar{z} + w_0.$$

As a result, the corresponding Onsager algebra turns out to be trivial since from the relations (5.11) it follows that all the higher generators of \mathfrak{DA} vanish in the limit $\alpha \rightarrow 0$ and the only

nontrivial generators are Z_0 , \bar{Z}_0 and B_1 . Similarly, except for J_n^1 , J_n^2 , all the integrals (3.7), vanish. Like in the 1D case, the resulting algebra obtained from (5.10) by the contraction $\alpha \rightarrow 0$ is a nilpotent algebra as well.

One notes that n -HSUSY can also be realized in the 2D systems with nontrivial Riemannian geometry. This case will be treated in detail elsewhere [31].

6 Nonlinear pseudo-supersymmetry

Till the moment we have discussed the Onsager algebra and its realizations in the context of nonlinear holomorphic supersymmetry with assumption of the involution (3.2). However, one can discard (3.2) and regard the generating elements of \mathfrak{DA} as Hermitian operators:

$$Z_0^\dagger = Z_0, \quad \bar{Z}_0^\dagger = \bar{Z}_0. \quad (6.1)$$

In this case the operators B_m become anti-Hermitian, and, as a consequence, the commuting charges (3.7) are not Hermitian operators any more. On the other hand, one can introduce the operator η such that

$$(\mathcal{J}_n^m)^\dagger = \eta \mathcal{J}_n^m \eta^{-1}. \quad (6.2)$$

In conventional matrix representation of the odd operators,

$$\theta^\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2), \quad (6.3)$$

η can be chosen, e.g., in the form $\eta = \sigma_1$. Then the relation (6.2) means that \mathcal{J}_n^m can be treated as self-conjugate operators with respect to the indefinite scalar product

$$(\psi_1|\psi_2) \equiv \langle \psi_1|\eta\psi_2 \rangle = \langle \psi_1\eta|\psi_2 \rangle, \quad (6.4)$$

where $\langle \psi_1|\psi_2 \rangle$ is the original positively definite scalar product. In other words, in the case of \mathfrak{DA} with generating elements (6.1) the operation of Hermitian conjugation can be changed for the pseudo-Hermitian conjugation,

$$O^\ddagger = \eta^{-1} O^\dagger \eta, \quad (6.5)$$

with respect to which the commuting charges are invariant:

$$\mathcal{J}_n^{m\ddagger} = \mathcal{J}_n^m. \quad (6.6)$$

Such a construction is very well known and appears, e.g., in the Dirac theory for spin-1/2 particles. Recently it emerged in the context of the PT -invariant systems with real spectrum [33, 18]. The property of the form (6.2) named the η -pseudo-Hermiticity is the necessary condition for having real spectra in the systems with the Hamiltonian satisfying the relation of the form (6.6) instead of to be a Hermitian operator [18]. There, the operator η was referred to as a Hermitian linear automorphism. In our case for the choice $\eta = \sigma_1$, the supercharges Q_n , \bar{Q}_n are pseudo-Hermitian operators, and, being nilpotent, they turn out to be similar in this sense to the Hermitian BRST and anti-BRST charges [34]. Hence, when

we treat the generating elements of \mathfrak{DA} as Hermitian operators, some sort of the nonlinear supersymmetry arises as well. Note, however, that unlike the present case, in the linear pseudo-supersymmetry [18] including the PT -symmetric supersymmetry [33] as a particular case, the nilpotent supercharges are mutually conjugate with respect to the indefinite scalar product (6.4).

It is worth noting that another possible way to overcome the non-Hermiticity of the commuting charges (3.7) for the Onsager algebra with Hermitian generating elements consists in performing the formal change $n \rightarrow in$. However, with such a prescription the nonlinear supersymmetric construction would be lost.

The examples of the systems with nonlinear supersymmetry realized in the form of pseudo-supersymmetry can be obtained within the framework of the spin chain models. First, we note that with taking into account the identification (4.1), the Hamiltonian of the Transverse Ising Chain given by the operator J_0 from Eq. (2.5) is constructed in terms of the Hermitian operators [24]

$$Z_0 = \sum_{i=1}^L \sigma_3(i) \sigma_3(i+1), \quad \bar{Z}_0 = \sum_{i=1}^L \sigma_1(i). \quad (6.7)$$

The system is defined on a periodic chain of length L with $\sigma_\alpha(i)$, $\alpha = 1, 2, 3$, being the Pauli matrices describing local spin on a site with number i . Since the operators (6.7) satisfy the Dolan-Grady relations, they can be used for the construction of the system with nonlinear holomorphic supersymmetry given in terms of the Hamiltonian $H_n = \mathcal{J}_n^1$ (3.9) and supercharges (3.10), where the Pauli matrices σ_a , $a = 1, 2, 3$, are supposed to be commuting with the matrices $\sigma_\alpha(i)$. Such a Hamiltonian is η -pseudo-Hermitian with $\eta = \sigma_1$. The number of independent commuting charges \mathcal{J}_n^m in such a supersymmetric model is the same as for the linear set (2.5) of the integrals corresponding to the Transverse Ising Model since they are based on the same representation of the Onsager algebra.

In the same way one can construct the nonlinear pseudo-supersymmetry proceeding from the Hermitian generating elements

$$Z_0 = \sum_{i=1}^L \sigma_3(i) \sigma_3(i+1), \quad \bar{Z}_0 = \sum_{i=1}^L \sigma_1(i) \sigma_1(i+1). \quad (6.8)$$

The operators (6.8) satisfy the Dolan-Grady relations and correspond to the X-Y model [23] like the operators (6.7) correspond to the Transverse Ising Model.

One more representation of the Dolan-Grady relations can be found in Z_N spin models [27] generalizing the Ising Model. In this case the Hermitian generating elements are

$$\bar{Z}_0 = \sum_{i=1}^L \sum_{n=1}^{N-1} \frac{X_i^n}{1 - \omega^{-n}}, \quad Z_0 = \sum_{i=1}^L \sum_{n=1}^{N-1} \frac{P_i^n P_{i+1}^{N-n}}{1 - \omega^{-n}}, \quad (6.9)$$

where $N = 2, 3, \dots$, and X_i , P_i are the local Z_N spin operators satisfying the relations $[X_i, X_j] = [P_i, P_j] = 0$, $P_i X_j = \omega^{\delta_{ij}} X_j P_i$, $P_i^N = X_i^N = \mathbb{1}$, $\omega = \exp(2\pi i/N)$. In the same line as in the previous two cases, one can construct the system with the pseudo-Hermitian Hamiltonian possessing the nonlinear supersymmetry.

It is worth noting that for all these spin models, one can construct the generating elements of the second generation, $Z_0^{(1)}$, $\bar{Z}_0^{(1)}$, and so on till some number defined by the length L of the periodic chain.

The described construction of the nonlinear pseudo-supersymmetry with some modification can be applied also to the quantum mechanical systems discussed in the previous section. As a result, one can obtain some quasi-exactly solvable systems which appear in the context of the PT -symmetric quantum mechanics. Let us obtain one such a family of the 1D quantum mechanical systems proceeding from the generators (5.1) with the quadratic superpotential (5.6). For such systems, unlike the n -supersymmetric class of systems with exponential form of the superpotential (5.2), the corresponding supersymmetry is always spontaneously broken when $w_2 \neq 0$ [4], while the case $w_2 = 0$ corresponds to the superoscillator with nonlinear supersymmetry to be the exact symmetry of the system [2]. The zero modes of the supercharge Q_n with the superpotential (5.6) have the following leading factor for $|x| \rightarrow \infty$:

$$\psi_0 \sim e^{-\frac{w_2}{3}x^3 - \frac{w_1}{2}x^2 - w_0x}. \quad (6.10)$$

Hence, such functions are not normalizable for $w_2 \neq 0$. Note, however, that if the parameter w_2 is pure imaginary and $w_1 > 0$, then the wave functions of the form (6.10) are normalizable. But in this case the Hamiltonian (2.2) is not Hermitian. If w_0 is also a pure imaginary number, there exists an operator η such that the Hamiltonian (2.2) is an η -pseudo-Hermitian operator. Indeed, let us put $\eta = P$, where P is the parity (reflection) operator, $PxP = -x$, $P^2 = 1$. Then, with respect to the pseudo-Hermitian conjugation (6.5), the operators (5.1) are transformed as $Z^\dagger = -\bar{Z}$, $\bar{Z}^\dagger = -Z$. As a consequence, the Hamiltonian (2.2) is pseudo-Hermitian. Thus, in this case we have the system with the nonlinear supersymmetry generated by the supercharges (3.10) having the properties $Q_n^\dagger = (-1)^n \bar{Q}_n$, $\bar{Q}_n^\dagger = (-1)^n Q_n$. One notes that although the parameters w_0, w_2 are pure imaginary, the anticommutator of the supercharges is a polynomial in the Hamiltonian with real coefficients. This property is in accordance with the generic property of any η -pseudo-Hermitian system for which the spectrum consists of real and/or complex conjugate pairs of numbers [18].

Rescaling the variable x , the obtained system given by the η -pseudo-Hermitian Hamiltonian and possessing nonlinear supersymmetry can be related to the three-parameter class of quasi-exactly solvable systems with quartic polynomial potentials discussed in Ref. [19] within the framework of the PT -symmetric quantum mechanics. One of the parameters taking there natural values, in our construction corresponds to the parameter n defining the order of the nonlinear supersymmetry.

7 Discussion and outlook

Let us summarize briefly the obtained results and discuss some open problems that deserve further attention.

- It was ascertained that the Onsager algebra associated with the Dolan-Grady relations underlies the nonlinear holomorphic supersymmetry.

The Onsager algebra arisen in the framework of the nonlinear holomorphic supersymmetry is endowed with a natural Hermitian conjugation. This involution is different from that associated with the Onsager algebra usually discussed in the context of integrable models with the infinite set of conserved charges constructed by Dolan and Grady. This is the main peculiarity of our treatment of the Onsager algebra.

- The contracted Onsager algebra corresponding to the contracted Dolan-Grady relations was introduced.

As far as we know such an algebra was not discussed earlier. This algebra has the properties similar to those of the original Onsager algebra but its structure is much more simple. Besides, for some systems the Dolan-Grady relations can be realized in the contracted form only. The examples of such systems are given by the superoscillator with nonlinear superalgebra [2], the single-mode parafermions [3], a charged spin-1/2 particle with gyromagnetic ratio $g = 2n$ subjected to an external magnetic field and moving in 2D space with non-trivial Riemannian geometry [31].

- The new infinite set of the commuting charges associated with the Onsager algebra was constructed.

Unlike the set (2.5) found by Dolan and Grady, in the new set (3.7) the charges have the terms quadratic in the generators of the Onsager algebra. The operators contain an arbitrary real parameter that can be interpreted as a coupling constant. We have shown that n -HSUSY is associated with this system for the coupling constant to be integer. In other words, for $n \in \mathbb{Z}$ there exists an intertwining operator that relates the spectra of the operators J_n^m and J_{-n}^m for any $m \in \mathbb{N}$.

- The notion of the nonlinear holomorphic supersymmetry was extended to the case of nonlinear pseudo-supersymmetry.

It seems that this construction could be useful for investigation of different PT -symmetric and other systems with pseudo-Hermitian Hamiltonians, the interest to which has considerably grown recently [18, 33].

In the context of the Onsager algebra construction, we clarified also the nature of the known 1D and 2D systems with nonlinear holomorphic supersymmetry [4, 6].

It is worth noting that the Onsager algebra admits a reformulation as a Poisson algebra of some mechanical system (see Ref. [35] for some classical realization of the Dolan-Grady relations). A priori, such a corresponding system may also possess the infinite set of commuting integrals, which could be obtained from (3.7) by the formal change $\{\bar{Z}, Z\} \rightarrow 2\bar{Z}Z$. Besides, the contracted Onsager algebra can be represented as a Poisson algebra of a continuous symplectic system. It would be interesting to find an explicit nontrivial example of such a classical system. If such hypothetical system realizes the infinite dimensional contracted Onsager algebra with linearly independent generators, then this system would be integrable due to existence of the infinite set of commuting integrals of motion. We hope that further investigation will shed light on this problem.

Acknowledgements

The work was supported by the grants 1010073 and 3000006 from FONDECYT (Chile) and by DY CIT (USACH). M.P. thanks A. Zhedanov for a very useful communication.

References

- [1] A.A. Andrianov, M.V. Ioffe and V.P. Spiridonov, Phys. Lett. A **174** (1993) 273-279, hep-th/9303005.
- [2] M. Plyushchay, Int. J. Mod. Phys. A **23** (2000) 3679-3698, hep-th/9903130.
- [3] S. Klishevich and M. Plyushchay, Mod. Phys. Lett. A **14** (1999) 2739-2752, hep-th/9905149.
- [4] S. Klishevich and M. Plyushchay, Nucl. Phys. B **606** (2001) 583-612, hep-th/0012023.
- [5] H. Aoyama, M. Sato and T. Tanaka, Phys. Lett. B **503** (2001) 423, quant-ph/0012065.
- [6] S. Klishevich and M. Plyushchay, Nucl. Phys. B **616** (2001) 403-418, hep-th/0105135.
- [7] P. Dorey, C. Dunning and R. Tateo, J. Phys. A **34** (2001) 5679, hep-th/0103051.
- [8] E. Witten, Nucl. Phys. B **188** (1981) 513.
- [9] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251** (1995) 267-385, hep-th/9405029.
- [10] V.V. Ulyanov and O.B. Zaslavskii, Zh. Eksp. Teor. Fiz., **87** (1984) 1724-1733.
- [11] A. Turbiner, Comm. Math. Phys. **118** (1988) 467.
- [12] M.A. Shifman, Int. J. Mod. Phys. A **4** (1989) 2897.
- [13] A. Ushveridze, Quasi-exactly solvable models in quantum mechanics, (IOP Publishing, Bristol, 1994).
- [14] F. Finkel, A. Gonzalez-Lopez, N. Kamran, P.J. Olver and M.A. Rodriguez, Lie algebras of differential operators and partial integrability, hep-th/9603139.
- [15] C.M. Bender and G.V. Dunne, J. Math. Phys. **37** (1996) 6-11, hep-th/9511138.
- [16] H. Aoyama, N. Nakayama, M. Sato and T. Tanaka, Phys. Lett. B **519** (2001) 260, hep-th/0107048.
- [17] For a short review see also S. M. Klishevich and M. S. Plyushchay, Nonlinear Supersymmetry, to appear in Proceedings of "Symmetry in Nonlinear Mathematical Physics", July 9-15, 2001, Kiev, Ukraine, hep-th/0112125.
- [18] A. Mostafazadeh, Pseudo-Hermiticity versus PT symmetry: The structure responsible for the reality of the spectrum of a non-Hermitian Hamiltonian, math-ph/0107001.

- [19] C. M. Bender and S. Boettcher, J. Phys. A: Math. Gen. **27** (1998) L273, physics/9801007.
- [20] E. Ragoucy and P. Sorba, Commun. Math. Phys. **203** (1999) 551; hep-th/9803243.
- [21] K. Schoutens, Phys. Lett. B **331** (1994) 335; hep-th/9401154.
- [22] De J. Boer, F. Harmsze, T. Tjin, Phys. Rep. **272** (1996) 139-214, hep-th/9503161.
- [23] L. Dolan and M. Grady, Phys. Rev. D **25** (1982) 1587.
- [24] L. Onsager, Phys. Rev. **65** (1944) 117.
- [25] B. Davies, J. Phys. A: Math. Gen. **23** (1990) 2245.
- [26] B. Davies, J. Math. Phys. **32** (1991) 2945.
- [27] G. von Gehlen and V. Rittenberg, Nucl. Phys. B **257** (1985) 351.
- [28] R. J. Baxter, Annals Phys. **76** (1973) 1.
- [29] R. J. Baxter, Annals Phys. **76** (1973) 25.
- [30] R. J. Baxter, Annals Phys. **76** (1973) 48.
- [31] S. Klishevich and M. Plyushchay, in preparation.
- [32] E. Date and S.S. Roan, J. Phys. A: Math. Gen. **33** (2000) 3275; math.QA/9911018.
- [33] M. Znojil, F. Cannata, B. Bagchi and R. Roychoudhury, Phys. Lett. B **483** (2000) 284, hep-th/0003277.
- [34] M. Henneaux and C. Teitelboim, Quantization of gauge systems, (Princeton University Press, 1992).
- [35] A. Zhedanov and A. Korovnichenko, “Leonard pairs” in classical mechanics, contribution to “Symmetry in Nonlinear Mathematical Physics”, July 9-15, 2001, Kiev, Ukraine.